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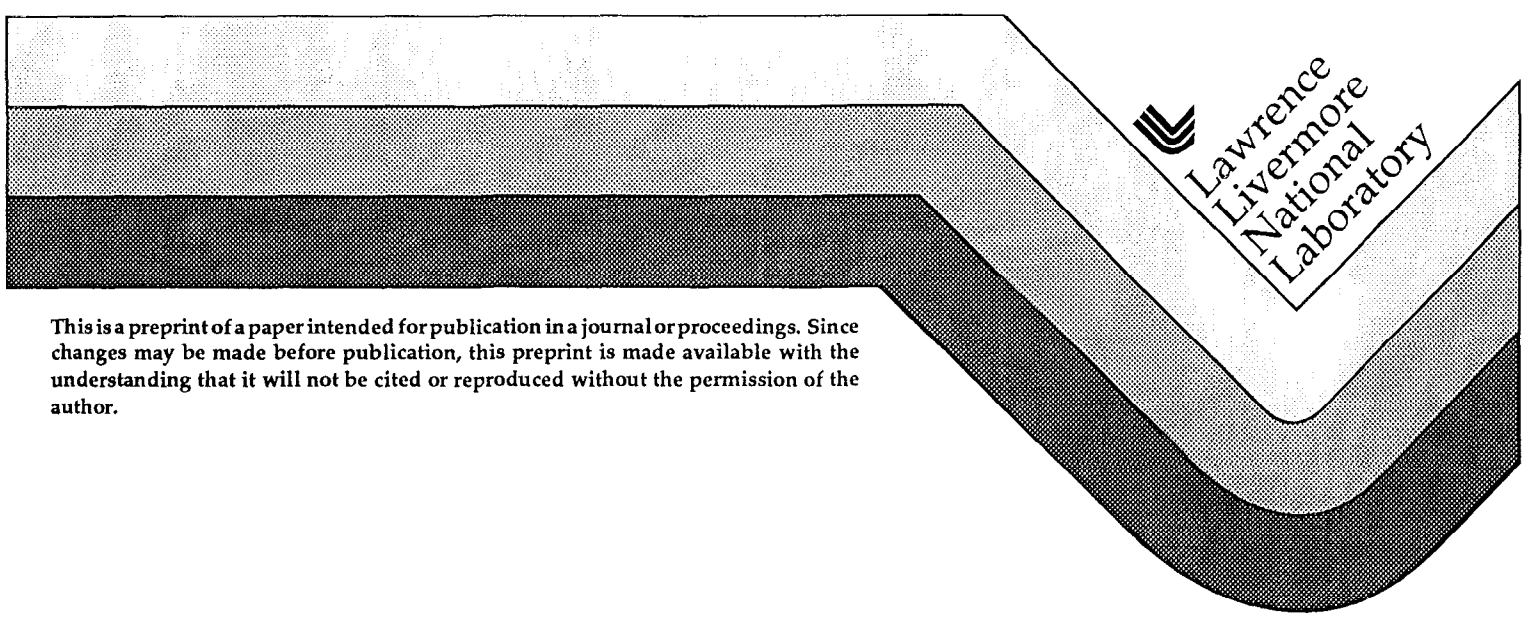

Solution of Capacitance Systems using Incomplete Cholesky Fixed Point Iteration

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Abstract

Application of the Galerkin finite element method to the electromagnetic vector wave equation yields an implicit system of equations that must be evolved in time The left hand matrix has units of capacitance and is analogous to the mass matrix in continuum mechanics In this letter we point out the interesting fact that for a Cartesian grid the Cholesky decomposition of the capacitance matrix has the same sparsity as the original matrix, i e there is no zero-fill during the course of the Cholesky decomposition Therefore an iterative method using the incomplete Cholesky decomposition as a preconditioner is quite efficient for nearly orthogonal quadrilateral or hexahedral grids

1 Introduction

The classic nodal finite element method has been shown to be an accurate and robust method for solving electrostatic problems However the use of nodal finite elements for fully electromagnetic problems has been problematic for several reasons First, the standard Lagrange nodal finite element does not allow jump discontinuities of fields across material interfaces Second, the use of nodal finite elements can lead to spurious, non-physical solutions For these reasons, vector finite element methods which employ the recently developed class of elements known as edge, Nedelec, or $H(curl)$ elements [1, 2] have become quite popular This paper is concerned with the solution of systems of equations that arise when using the linear vector finite element method for the electromagnetic vector wave equation The simple Incomplete Cholesky Fixed Point (ICFP) iteration is investigated as an iterative method. It is shown that this method is quite efficient for general grids, and in fact this method converges to the exact solution in a single iteration for Cartesian grids if the degrees-of-freedom are ordred properly

2 Galerkin Formulation

We are concerned with the computer simulation of time dependent electromagnetic fields in a generic volume Ω. There is no free charge in the volume An appropriate PDE is the vector wave equation for the electric field $\vec{E}$

$$\epsilon \frac{\partial^2}{\partial t^2}\vec{E} = -\nabla \times \mu^{-1}\nabla \times \vec{E} - \frac{\partial}{\partial t}\vec{J} \text{ in } \Omega \tag{1}$$

For simplicity it is assumed that the dielectric permittivity ϵ and the magnetic permeability μ are constant scalars. The electric field on the boundary Γ is specified by

$$\hat{n} \times \vec{E} = \vec{E}_{bc} \text{ on } \Gamma, \tag{2}$$

and the two initial conditions

$$\vec{E}(t=0) = \vec{E}_{ic} \text{ in } \Omega, \tag{3}$$

$$\frac{\partial}{\partial t}\vec{E}(t=0) = \frac{\partial}{\partial t}\vec{E}_{ic} \text{ in } \Omega, \tag{4}$$

complete the description of the PDE Typically the initial conditions are zero and the problem is driven by either the time dependent current source $\vec{J}$ or the time dependent boundary condition $\vec{E}_{bc}$

The variational form of (1) is: find $\vec{E} \in H(curl)$ that satisfies

$$\frac{\partial^2}{\partial t^2}\left(\epsilon\vec{E}, \vec{E}^*\right) = \left(\mu^{-1}\nabla \times \vec{E}, \nabla \times \vec{E}^*\right) - \frac{\partial}{\partial t}\left(\vec{J}, \vec{E}^*\right) \tag{5}$$

for all $\vec{E}^* \in H_0(curl)$, where

$$(\vec{u}, \vec{v}) = \int_\Omega \vec{v} \ \ \vec{u} d\Omega, \tag{6}$$

and

$$H_0(curl) = \{\vec{v} : \vec{v} \in H(curl), \hat{n} \times \vec{v} = 0\} \tag{7}$$

In the finite element solution of (5) the space $H(curl)$ is approximated by a finite dimension subspace $W^h \subset H(curl)$ defined on a mesh, yielding a system of ODE's

$$A\frac{\partial^2}{\partial t^2}\tilde{e} = C\tilde{e} + \tilde{s}. \tag{8}$$

The variable $\tilde{e}$ is the array of degrees-of-freedom and the variable $\tilde{s}$ is the array of source terms, which includes contributions from both the independent current source $\vec{J}$ and the boundary condition $\vec{E}_{bc}$ The matrix A is a symmetric positive definite matrix, with units of capacitance, which resembles the mass matrix of continuum mechanics The matrices A and C are given by

$$A_{ij} = \left(\epsilon\vec{W}_i, \vec{W}_j\right), \tag{9}$$

$$C_{ij} = \left(\mu^{-1}\nabla \times \vec{W}_i, \nabla \times \vec{W}_j\right), \tag{10}$$

where $\vec{W}_i$ is the basis function associated with edge i The vector basis functions $\vec{W}_i$ are well known and will not be derived here The properties of $\vec{W}_i$ relevant to this letter are: 1) $\left(\vec{W}_i, \vec{W}_j\right) = 0$ if edges i and j do not share an element, and 2) $\left(\vec{W}_i, \vec{W}_j\right) \propto \cos\theta$, where θ is the angle between edges i and j

3 Incomplete Cholesky Fixed Point Iteration

From now on we are concerned with the solution of the system $Ax = b$ Let the capacitance matrix be decomposed as

$$A = M - N \tag{11}$$

The matrix M is the preconditioner. The fixed point iteration is then

$$Mx^{k+1} = Nx^k + b. \tag{12}$$

This iteration is known to converge if and only if

$$\rho\left(M^{-1}N\right) < 1, \tag{13}$$

where $\rho(A)$ denotes the spectral radius of A Since A is symmetric positive definite it has a Cholesky decomposition $A = LL^T$ where L is a lower triangular matrix The Incomplete Cholesky Fixed Point (ICFP) iteration is defined by using

$$M = \tilde{L}\tilde{L}^T, \tag{14}$$

where $\tilde{L}\tilde{L}^T$ is the incomplete Cholesky decomposition of A The Cholesky decomposition defined in [3] is shown in Fig 1 for convienence The incomplete Cholesky decomposition is a modification of the above algorithm such that line 8 is executed only if $A(i,j) = 0$. Therefore the incomplete Cholesky decomposition has the same sparsity pattern as the original matrix. If the matrix A has at most m non-zero entries per row, then the ICFP iteration requires approximatly mN_e floating point operations, where N_e is the number of internal edges in the grid The number of ICFP iterations required depends upon how effective the incomplete Cholesky decomposition is as a preconditioner

For a 2D or 3D Cartesian grid there is no zero-fill during the course of the Cholesky decomposition, hence the ICFP gives the exact solution in a single iteration. Consider line 8 of Fig 1 Assume that $A(i,j) = 0$, which means that there is no interaction between edges i and j. There will be

```
1  for k = 1:n
2    A(k,k) = sqrt(A(k,k))
3    for i = k+1:n
4      A(i,k) = A(i,k)/A(k,k)
5    end
6    for j = k+1:n
7      for i = j:n
8        A(i,j) = A(i,j) - A(i,k)*A(j,k)
9      end
10   end
11 end
```

Figure 1: Cholesky decomposition algorithm for n by n matrix A

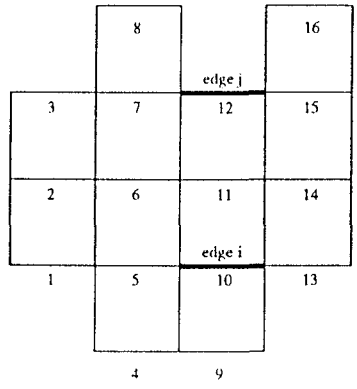

Figure 2. Numbering scheme used to eliminate zero-fill

zero fill only if there is another edge k, with $k < j \leq i$, that interacts with both edge i and edge j Numbering the edges sequentially precludes this possibility This numbering scheme is illustrated in Fig 2

For a distorted quadrilateral or hexahedral grid, there is zero-fill during the course of the Cholesky decomposition, hence the ICFP is not exact. Again consider line 8 of Fig. 1, and assume that $A(i,j) = 0$. For a distorted grid there will be some edge k, with $k < j \leq i$, that interacts with both edge i and edge j This is illustrated in Fig 3 However, since $A(i,k) * A(j,k) \propto (\cos\alpha)^2$, the terms that are ignored are quite small for grids that are not too distorted In [4] it was shown that for the case of a triangular grid, a grid composed of equalateral triangles was ideal in the sense that A had minimum condition number The result here is similar, except that the condition number of $M^{-1}A$ is actually unity for the ideal grid

As an example, consider the grid shown in Fig. 4. This grid has 1000 nodes and 729 distorted hexahedral cells The capacitance matrix has dimension 1728, the number of internal edges Using an initial iterate of $x_0 = 0$, the ICFP method required only 5 iterations to achieve a L_2 residual of 10^{-9} As another example, consider the spherical grid shown in Fig 5 This grid has 2273 nodes and 2048 hexahedral cells The capacitance matrix has dimension 5792 For this grid the ICFP method required 7 iterations to achieve a L_2 residual of 10^{-9} This grid required more iterations not because it was larger, but because the minimum angle between adjacent edges was smaller This same number of iterations was required for grids with over 180,000 edges

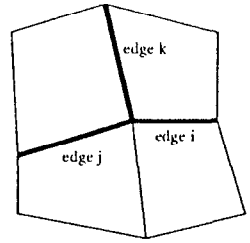

Figure 3 Zero-fill occurs for distorted grids

4 Summary

The Incomplete Cholesky Fixed Point iteration is suggested as a method for solving capacitance systems that arise in the vector finite element solution of electromagnetic vector wave equation This method has the properly that it converges to the exact solution in a single iteration for Cartesian grids. For hexahedral grids that are not too distorted the method requires between 5 and 10 iterations to achieve an L_2 residual if 10^{-9}, regardless of problem size

References

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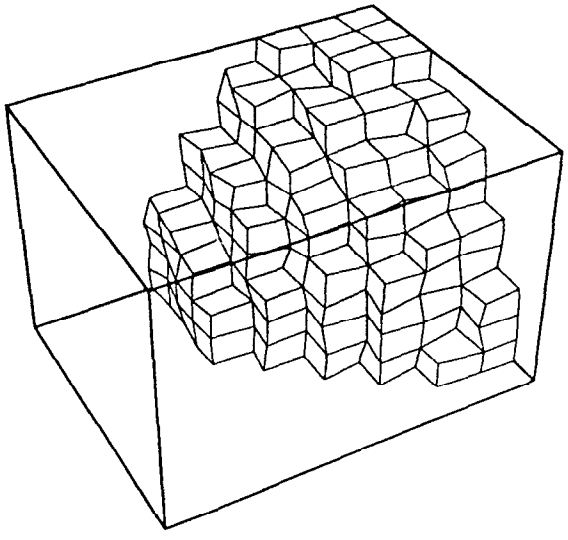

Figure 4: Distorted hexahedral grid of rectangular region

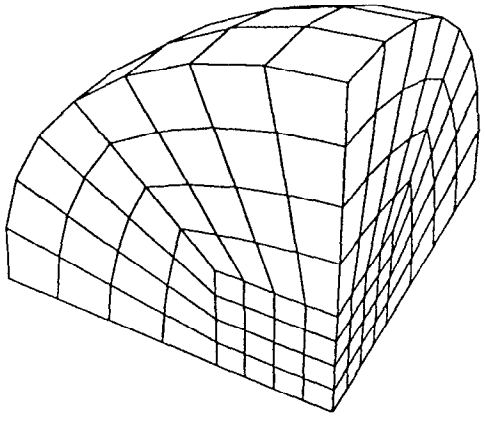

Figure 5: Hexahedral grid of spherical region